

Higher dimensional Robinson–Trautman spacetimes sourced by p -forms: static and radiating black holes

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We summarize results about Robinson–Trautman spacetimes in the presence of an aligned p -form Maxwell field and an arbitrary cosmological constant in $n \geq 4$ dimensions. While in *odd* dimensions the solutions reduce to static black holes dressed with an electric and a magnetic field (with an Einstein space horizon), in *even* dimensions $2p = n$ they may also describe black holes gaining (or losing) mass by receiving (or emitting) electromagnetic radiation. The Weyl type of the spacetimes is also briefly discussed in all the possible cases.

1. Introduction

The Einstein–Maxwell theory can be generalized by considering p -form fields \mathbf{F} coupled to gravity for an arbitrary $0 < p < n$ (n is the number of spacetime dimensions), cf., e.g.,¹. This is of particular interest when $n > 4$. The corresponding Maxwell equations, in the source-free case (to which we will restrict hereafter), read

$$(\sqrt{-g} F^{\mu\alpha_1\ldots\alpha_{p-1}})_{,\mu} = 0, \quad F_{[\alpha_1\ldots\alpha_p,\mu]} = 0. \quad (1)$$

The back-reaction of \mathbf{F} enters Einstein’s equations as

$$R_{\mu\nu} = \frac{2}{n-2} \Lambda g_{\mu\nu} + \kappa_0 \left[F_{\mu\alpha_1\ldots\alpha_{p-1}} F_{\nu}{}^{\alpha_1\ldots\alpha_{p-1}} - \frac{p-1}{p(n-2)} g_{\mu\nu} F^2 \right], \quad (2)$$

where $F^2 = F_{\alpha_1\ldots\alpha_p} F^{\alpha_1\ldots\alpha_p}$, κ_0 is a coupling constant and Λ is the cosmological constant.

Electric static black holes in the standard case $p = 2$ were studied already in the early paper², and a magnetic “charge” (in even dimensions) was added in³. More recently, it has been shown that asymptotically flat static black holes cannot couple to electric p -form fields when $(n+1)/2 \leq p \leq n-1$ (and thus do not possess dipole hair)⁴ and that, for any $p > 2$, static perturbations of the vacuum Schwarzschild–Tangherlini metric do not exist^{5,6}. Subsequently, a large family of p -form black holes (with various possible horizon geometries and asymptotia) has been constructed in⁷ (see³ for related earlier work in the case $p = 2$). In addition, results of⁸ showed that electromagnetic radiation may have properties different from those of standard $n = 4$, $p = 2$ electrovac general relativity (except for $2p = n$), as confirmed in⁹ in the case of test fields.

Our recent work¹⁰, to be summarized here, considered exact solutions of p -form Einstein-Maxwell gravity in the broader context of Robinson-Trautman spacetimes, which allow also for radiative (non-static) solutions. The Robinson-Trautman class is defined by the existence of a geodesic, shear-free, twist-free but expanding null vector field \mathbf{k} and was first studied in $n = 4$ general relativity¹¹ (see also the reviews in^{12,13}). This was extended to an arbitrary n in¹⁴ and further studied in^{3,15–18}. Some of the general results of^{14,18} have been summarized in Theorem 1 of Appendix A of¹⁰, which may be useful to recall here:

Theorem 1.1 (Robinson-Trautman spacetimes of aligned Ricci type II).

If a n -dimensional spacetime ($n \geq 4$) admits a non-twisting, non-shearing, expanding geodesic null vector field \mathbf{k} and the Ricci tensor is of aligned type II, adapted coordinates $(u, r, x^1, \dots, x^{n-2})$ can be chosen such that¹⁴

$$ds^2 = r^2 h_{ij} (dx^i + W^i du) (dx^j + W^j du) - 2 du dr - 2H du^2, \quad (3)$$

$$h_{ij} = h_{ij}(u, x), \quad W^i = \alpha^i(u, x) + r^{1-n} \beta^i(u, x), \quad (4)$$

$$\mathbf{k} = \partial_r, \quad \theta = 1/r, \quad (5)$$

where H is an arbitrary function of all coordinates. \mathbf{k} is automatically a WAND, such that the Weyl tensor is in general of aligned type I(b). It is a multiple WAND iff $\beta^i = 0$ ¹⁸, in which case the Weyl tensor is of aligned type II(d) (or more special). When $\beta^i = 0$, one can locally set $W^i = 0$ (after a coordinate transformation giving $\alpha^i = 0$)¹⁴. The Weyl type further specializes to II(bd) iff h_{ij} is an Einstein metric (with still $W^i = 0$)¹⁸.

Here and in the following, the vector field \mathbf{k} is the generator of null hypersurfaces $u = \text{const}$ such that $k_\mu dx^\mu = -du$, r is an affine parameter along \mathbf{k} , θ is its expansion scalar, and $x \equiv (x^i) \equiv (x^1, \dots, x^{n-2})$ are spatial coordinates on a “transverse” $(n-2)$ -dimensional Riemannian manifold. For certain calculations it may be useful to observe that $2H = g^{rr} = -g_{uu}$ and $W^i = g^{ri}$ (such that $W^i = 0 \Leftrightarrow g^{ri} = 0 \Leftrightarrow g_{ui} = 0$). Recall that in *vacuum* or with *aligned pure radiation* necessarily $\beta^i = 0$ and h_{ij} is Einstein for any $n \geq 4$, and the Weyl tensor further specializes to type D(bd) (possibly, D(bcd) or D(acd)) if $n > 4$, cf.¹⁴. If one relaxes the assumptions of the theorem by requiring only the aligned Ricci type I (i.e., $R_{rr} = 0$), one obtains the same form of the metric, except that the $W^i(u, r, x)$ are arbitrary functions¹⁴.

The above theorem includes, in particular, the case of Robinson-Trautman spacetimes in the presence of an aligned p -form \mathbf{F} , i.e., such that

$$F_{ri_1 \dots i_{p-1}} = 0. \quad (6)$$

The corresponding Einstein-Maxwell equations have been studied in¹⁰. As it turns out, one has to consider separately the case of a generic n (section 2), and the special case of an even $n = 2p$ (section 3). The case $p = 1$ is special in all dimensions and leads to static metrics with a non-Einstein transverse space, see¹⁰ for details.

2. Generic case $2p \neq n$ ($n > 4$): static black holes

The line-element is given by

$$ds^2 = r^2 h_{ij} dx^i dx^j - 2 du dr - 2H du^2, \quad (7)$$

where $h_{ij} = h_{ij}(x)$ is the metric of a Riemannian Einstein space of dimension $(n-2)$ and scalar curvature $\mathcal{R} = K(n-2)(n-3)$, and

$$2H = K - \frac{2\Lambda}{(n-1)(n-2)} r^2 - \frac{\mu}{r^{n-3}} + \frac{\kappa_0}{n-2} \left[\frac{p-1}{n+1-2p} \frac{\mathcal{E}^2}{r^{2(n-p-1)}} - \frac{1}{p(n-1-2p)} \frac{\mathcal{B}^2}{r^{2(p-1)}} \right] \quad (2p \neq n \pm 1), \quad (8)$$

where Λ , μ , \mathcal{E}^2 , \mathcal{B}^2 and $K = 0, \pm 1$ are constants. There is a Killing vector field ∂_u (so that the metric is static in regions where $H > 0$) and roots of $H(r)$ define Killing horizons (see also⁷). In the special cases $2p = n \pm 1$ (n odd), the second line of (8) should be replaced by appropriate logarithmic terms (which in fact vanish for $n = 5, 7$, see¹⁰ for details).

The corresponding ‘‘Coulombian’’ Maxwell field is given by

$$\mathbf{F} = \frac{1}{(p-2)!} \frac{1}{r^{n+2-2p}} e_{i_1 \dots i_{p-2}}(x) du \wedge dr \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{p-2}} + \frac{1}{p!} b_{i_1 \dots i_p}(x) dx^{i_1} \wedge \dots \wedge dx^{i_p}, \quad (9)$$

where $e_{i_1 \dots i_{p-2}}$ and $b_{i_1 \dots i_p}$ are harmonic forms (of respective rank $(p-2)$ and p) in the transverse geometry h_{ij} . These forms are further constrained by

$$\mathcal{E}_{ij}^2 = \frac{\mathcal{E}^2}{n-2} h_{ij} \quad (p \geq 3), \quad \mathcal{B}_{ij}^2 = \frac{\mathcal{B}^2}{n-2} h_{ij} \quad (p \leq n-2), \quad (10)$$

$$\text{where} \quad \mathcal{E}_{ij}^2 \equiv e_{ik_1 \dots k_{p-3}} e_j^{k_1 \dots k_{p-3}}, \quad \mathcal{E}^2 \equiv h^{ij} \mathcal{E}_{ij}^2 \quad (p \geq 3), \quad (11)$$

$$\mathcal{B}_{ij}^2 \equiv b_{ik_1 \dots k_{p-1}} b_j^{k_1 \dots k_{p-1}}, \quad \mathcal{B}^2 \equiv h^{ij} \mathcal{B}_{ij}^2 \quad (p \leq n-2). \quad (12)$$

Conditions (10) also impose severe restrictions on the Einstein metric h_{ij} ^{3,7,10}.

The above solutions are extensions to arbitrary p of the $p = 2$ ($n \neq 6$) solutions studied in³ (including, when $\mathcal{B}^2 = 0$, the $n > 4$ Reissner-Nordström solution found by Tangherlini²) and possess similar properties. They were previously obtained in⁷ (using a different method) and represent static black holes dressed with electric and magnetic Maxwell fields, at least for certain values of the parameters in (8). However, the metric h_{ij} in (7) cannot describe a round sphere¹⁰, except when $p = 2$ and $b_{i_1 i_2} = 0$ (or, dually, when $p = n-2$ and $e_{i_1 \dots i_{n-4}} = 0$), so that these black holes cannot have a spherical horizon and cannot be asymptotically flat (in agreement with⁴⁻⁶), except in the electric $p = 2$ (or magnetic $p = n-2$) Reissner-Nordström solution of². A flat and compact horizon metric h_{ij} is instead permitted (giving $K = 0$ in (8); then the harmonic forms $e_{i_1 \dots i_{p-2}}$ and $b_{i_1 \dots i_p}$ must be covariantly constant^{19,20}), thus allowing, e.g., for asymptotically locally (A)dS black holes (see also⁷). An additional ‘‘Vaidya-type’’ matter field representing pure radiation aligned with \mathbf{k} can easily be included by appropriate simple modifications^{3,7,10}.

The Weyl type of the spacetimes is D(bd) and $\mathbf{k} = \partial_r$, $\mathbf{l} = -\partial_u + H\partial_r$ are the (unique) double WANDs. These are manifestly also aligned null directions of the Maxwell field (9), which is thus also of type D. Also the Ricci tensor is of (aligned) type D. See^{2,3,7,10} for explicit examples with various values of n and p .

Let us just observe that more general solutions are permitted in the *exceptional cases* $2p = n \pm 2$ ($n \geq 6$, *even*), which are in general non-static and of Weyl type II(bd) – see¹⁰ for details.

3. Special case $2p = n$ (n even): black holes with electromagnetic radiation

Static black hole solutions clearly exist also for the special rank $2p = n$. However, there exists now also a new family of solutions which allows for $F_{uj_1 \dots j_{p-1}} \neq 0$. The Maxwell field is given by

$$\mathbf{F} = \frac{1}{(\frac{n}{2} - 2)!} \frac{1}{r^2} e_{i_1 \dots i_{p-2}} du \wedge dr \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{p-2}} + \frac{1}{(\frac{n}{2})!} b_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p} \\ + \frac{1}{2(\frac{n}{2} - 1)!} \left(-\frac{n-2}{r} e_{[i_2 \dots i_{p-1}, i_1]} + 2f_{i_1 \dots i_{p-1}} \right) du \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{p-1}}. \quad (13)$$

The forms $e_{i_1 \dots i_{p-2}}(u, x)$ and $b_{i_1 \dots i_p}(u, x)$ are generally not harmonic in the transverse space, but satisfy the “modified” Euclidean Maxwell equations in $(n-2)$ dimensions

$$(\sqrt{h} e^{ji_1 \dots i_{p-3}})_{,j} = 0 \quad (p \geq 3), \quad b_{[i_1 \dots i_p, j]} = 0, \quad (14)$$

$$(\sqrt{h} b^{kj_1 \dots j_{p-1}})_{,k} = \frac{1}{2}(n-2)\sqrt{h} h^{i_1 j_1} \dots h^{i_{p-1} j_{p-1}} e_{[i_2 \dots i_{p-1}, i_1]}. \quad (15)$$

In addition, they can depend on u via

$$b_{i_1 \dots i_p, u} = \frac{n}{2} f_{[i_2 \dots i_p, i_1]}, \quad (\sqrt{h} e^{i_1 \dots i_{p-2}})_{,u} = (\sqrt{h} f^{ki_1 \dots i_{p-2}})_{,k}. \quad (16)$$

The latter further tells us that the $(p-1)$ -form $f_{i_1 \dots i_{p-1}}(u, x)$ is also generically non-harmonic.

The line-element is given by (7) with

$$2H = K + \frac{2(\ln \sqrt{h})_{,u}}{n-2} r - \frac{2\Lambda}{(n-1)(n-2)} r^2 - \frac{\mu}{r^{n-3}} + \frac{\kappa_0}{2} \left(\mathcal{E}^2 + \frac{4}{n(n-2)} \mathcal{B}^2 \right) \frac{1}{r^{n-2}}, \quad (17)$$

where \mathcal{E}^2 , \mathcal{B}^2 and μ are generically functions of u and x , cf. the corresponding equations given in¹⁰. The metric $h_{ij}(u, x) = h^{1/(n-2)}(u, x) \gamma_{ij}(x)$ is again Einstein ($\gamma_{ij}(x)$ is unimodular), with scalar curvature $\mathcal{R} = K(n-2)(n-3)$ (where $K = 0, \pm 1$), with the constraint (recall (11), (12))

$$\frac{1}{4}(n-2)(n-4) \left(\mathcal{E}_{ij}^2 - \frac{\mathcal{E}^2}{n-2} h_{ij} \right) = \mathcal{B}_{ij}^2 - \frac{\mathcal{B}^2}{n-2} h_{ij}. \quad (18)$$

Considerable simplifications arise if h_{ij} is taken to be the metric of a compact space¹⁰. In particular, h_{ij} *cannot describe a round sphere*, and no asymptotically

flat spacetimes are thus to be found in this class of solutions. In general, the Weyl tensor is of type II(bd). The type D(bd) (or D(bcd)) is possible in special cases, but the types III, N and O are forbidden when $\mathbf{F} \neq 0$.

The Maxwell field (13) is in general of type II (aligned with \mathbf{k} by construction) and, in a parallelly transported frame adapted to \mathbf{k} , peels off as $\mathbf{F} = \frac{\mathbf{N}}{r^{\frac{n}{2}-1}} + \frac{\mathbf{II}}{r^{\frac{n}{2}}}$ (in agreement with test-field results⁹). It becomes of type N iff $e_{i_1 \dots i_{p-2}} = 0 = b_{i_1 \dots i_p}$, and of type D under the conditions given in¹⁰. When p is odd, a self-dual \mathbf{F} is possible in special cases.

For $n = 6 = 2p$, a simple example is given by

$$\begin{aligned} h_{ij} &= \delta_{ij}, & 2H &= -\frac{\Lambda}{10}r^2 - \frac{\mu(u)}{r^3}, & \mu(u) &= \mu_0 - \frac{\kappa_0}{2} \int \mathcal{F}^2 du, \\ F_{uij} &= f_{ij}(u). \end{aligned} \tag{19}$$

For $\Lambda < 0$ this spacetime represents the formation of asymptotically locally AdS black holes with electromagnetic radiation. By a rotation, one can always simplify the Maxwell field so as to only have non-zero components $F_{u12} = f_{12}(u)$, $F_{u34} = f_{34}(u)$, in which case \mathbf{F} is (anti-)self-dual when $f_{34}(u) = \mp f_{12}(u)$. Solution (19) is an extension of a solution given for $n = 4$ (for $\Lambda = 0$) in¹¹ and recently discussed in²¹.

See¹⁰ for more details and other examples.

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